Distributed Hypothesis Testing

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We model a network composed of n agents as a graph $G = \{V, E\}$. $V = \{1, 2, \dots, n\}$ is the set of vertices representing the agents. $E \subseteq V \times V$ is the set of edges. $(i, j) \in E$ if and only if sensor i and j can communicate directly with each other. We will always assume that G is undirected, i.e. $(i, j) \in E$ if and only if $(j, i) \in E$. We further assume that there is no self loop, i.e., $(i, i) \notin E$.

At each time, each sensor make an i.i.d. measurement $y_i(k)$. Consider the following two hypothesis:

H0 : $y_i(k) \sim \mathcal{N}(0, 1)$.

H1 : $y_i(k) \sim \mathcal{N}(1, 1)$.

We assume that each hypothesis is true with 0.5 probability.

1 Centralized Detector

The optimal centralized detector is a Naive Bayes detector. Define the average to be

$$\alpha(k) = \frac{1}{n(k+1)} \sum_{t=0}^{k} \sum_{i=1}^{n} y_i(k).$$

Hence, the centralized detector is

$$f(\alpha(k)) = \begin{cases} 0 & \text{if } \alpha(k) \leq 0.5 \\ 1 & \text{if } \alpha(k) > 0.5 \end{cases}$$

Define the probability of error of such detector to be $P_c(k)$, then

$$P_c(k) = 0.5P(\alpha(k) \leq 0.5|H_1) + 0.5P(\alpha(k) > 0.5|H_0) = P(\alpha(k) > 0.5|H_0)$$

We use large deviation theory to characterize $P_c(k)$. Suppose $x_i(k) \sim \mathcal{N}(0, 1)$, then the moment generating function is given by

$$M(\theta) = \int_{-\infty}^{\infty} \exp(\theta t) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt = \exp\left(\frac{\theta^2}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(t-\theta)^2}{2}\right) dt = \exp\left(\frac{\theta^2}{2}\right).$$

Hence, the log-moment generating function is

$$\Lambda(\theta) = \log M(\theta) = \frac{\theta^2}{2},$$

and

$$I(0.5) = \sup_\theta 0.5\theta - \Lambda(\theta) = \frac{1}{8}.$$

Hence, we know that $P_c(k) \sim \exp(-nk/8)$.

2 Distributed Detection

Let A be a consensus matrix that is compatible with the topology G, such that

- A has an eigenvalue of 1 and all the other eigenvalues of A are strictly inside the unit disk.
- $\mathbf{1}$ is both a left and right eigenvector of A.

Define $J = \mathbf{1}\mathbf{1}^T/n$, then $A^k \to J$ as $k \to \infty$.

Define the state of sensor i at time k to be $x_i(k)$. The sensor update equation can be written as

$$x_i(k)^+ = \frac{k}{k+1} x_i(k) + \frac{1}{k+1} y_i(k),$$
$$x_i(k+1) = a_{ii} x_i(k)^+ + \sum_{j \in N_i} a_{ij} x_j(k)^+.$$

Hence,

$$x(k)^+ = \frac{k}{k+1} x(k) + \frac{1}{k+1} y(k),$$
$$x(k+1) = A x(k)^+.$$

Let us define

$$\bar{x}(k+1) = Jx(k+1) = \alpha(k)\mathbf{1}.$$

For each sensor, it implements a detector f_i, which is defined as

$$f_i(x_i(k)) = \begin{cases} 0 & \text{if } x_i(k) \le 0.5 \\ 1 & \text{if } x_i(k) > 0.5 \end{cases}$$

Denote the probability of error for each individual detector as $P_i(k)$. By symmetry, we know that

$$P_i(k) = P(x_i(k) \ge 0.5 | H_0).$$

Clearly $x_i(k) < 0.5$ if $\alpha(k-1) < 0.5 - \delta$ and $x_i(k) - \alpha(k-1) < \delta$, for any $\delta > 0$. Hence,

$$P_i(k) = P(\alpha(k-1) \ge 0.5 - \delta | H_0) + P(x_i(k) - \alpha(k-1) \ge \delta | H_0).$$

For the first probability, we know that

$$I(0.5-\delta) = \sup_{\theta}(0.5-\delta)\theta - 0.5\theta^2 = 0.125+\varepsilon,$$

where $\varepsilon \to 0$ when $\delta \to 0$. Hence, $P(\alpha(k-1) \geq 0.5-\delta|H_0) \sim \exp(-(0.125+\varepsilon)nk)$.

Now let us look at $k(x_i(k)-\alpha(k-1))$. We know that

$$k(x(k)-\bar{x}(k)) = \left[(A-J)y(k-1) + (A-J)^2 y(k-2) + \cdots + (A-J)^k y(0)\right].$$

Hence, under hypothesis H_0, $k(x(k)-\bar{x}(k))$ is Gaussian distributed with zero mean and with covariance:

$$(A-J)(A-J)^T + \cdots + (A-J)^{k+1}\left((A-J)^{k+1}\right)^T \leq M.$$

Therefore, $k(x_i(k)-\alpha(k-1))$ is Gaussian distributed with zero mean and a bounded variance.

$$P(x_i(k)-\alpha(k-1) \geq \delta|H_0) = \frac{1}{\sqrt{2\pi}}\int_{k\delta/\sigma(k)}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt,$$

where $\sigma(k)$ is the standard deviation of $k(x_i(k)-\alpha(k-1))$.

For any $x > 0$, we have that

$$\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt \leq \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} \frac{t}{x}\exp\left(-\frac{t^2}{2}\right) dt \leq \frac{1}{x\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right).$$

Therefore, for large enough k

$$P(x_i(k)-\alpha(k-1) \geq \delta|H_0) \leq \exp\left(-\frac{1}{2M_{ii}}k^2\delta^2\right).$$

Hence, one can prove that for any $\varepsilon > 0$, and large enough k,

$$P_i(k) \leq \exp\left(-(0.125+\varepsilon)nk\right).$$

On the other hand, $P_i(k) \geq P_c(k)$, which implies that

$$P_i(k) \geq \exp\left(-(0.125-\varepsilon)nk\right).$$

Hence, $n/8$ is the rate function for $P_i(k)$.