Variants of Average Consensus

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1 Finite Time Average Consensus

Consider the following update equation:

$$x(k+1) = (I - \alpha(k)L)x(k), \tag{1}$$

We know if we fix $0 < \alpha(k) < 2/\lambda_n(L)$, then

$$\prod_{k=0}^{\infty}(I - \alpha(k)L) = J = \frac{\mathbf{1}\mathbf{1}^T}{n}.$$

Assume that $L = U\Lambda U^T$, then

$$\prod_{k=0}^{n-2} = U \begin{bmatrix} 1 & & & \\ & f(\lambda_2(L)) & & \\ & & \ddots & \\ & & & f(\lambda_n(L)) \end{bmatrix} U^T,$$

where $f(x)$ is an $n-1$th degree polynomial of the following form:

$$f(x) = \prod_{k=0}^{n-2}(1 - \alpha(k)x).$$

Hence, if we choose $\alpha(0) = 1/\lambda_2(L)$,...,$\alpha(n-2) = 1/\lambda_n(L)$, then $f(\lambda_i(L)) = 0$, for any $i = 2, \dots, n$. Thus, we can reach consensus in $n-1$ steps.

In general, if we do not know all the eigenvalues of L, but suppose that we know $\lambda_i(L) \in [a, b]$, for all $i = 2, \dots, n$. Further assume that we can use a periodic $\alpha(k)$, with $\alpha(k+T) = \alpha(k)$, then the problem becomes finding a Tth polynomial $f(x)$, such that

$$\begin{array}{ll} \underset{f(x)}{\text{minimize}} & \max_{x\in[a,b]} |f(x)| \\ \text{subject to} & f(0) = 1 \\ & f(x) \text{ is a } T\text{th degree polynomial} \end{array}$$

If $T = 1$, then the best function is $f(x) = 1 - \frac{2}{a+b}x$.

For higher T, $f(x)$ will be a scaled and shifted Chebyshev polynomial, which gives

$$\alpha(k) = \frac{b-a}{2}\cos\left(\frac{2k+1}{2T}\pi\right) + \frac{a+b}{2}, k = 0, \ldots, T-1.$$

2 Consensus with Noise

We use the following consensus scheme:

$$x(k+1) = (I - \alpha L)x(k),$$

where L is the Laplacian matrix and $\alpha > 0$. Hence,

$$x_i(k+1) = (1 - d_i\alpha)x_i(k) + \alpha \sum_{j\in\mathcal{N}_i} x_j(k), \tag{2}$$

where $\mathcal{N}_i$ is the set of the neighboring node of i and $d_i = |\mathcal{N}_i|$ is the degree of node i.

Notice that $x_j(k)$ in (2) is the message received by node i from node j. Now consider that instead of receiving $x_j(k)$, the node is receives $z_{ij}(k)$, which is a noisy version of $x_j(k)$:

$$z_{ij}(k) = x_j(k) + w_{ij}(k).$$

Hence, (2) becomes:

$$x_i(k+1) = (1 - d_i\alpha)x_i(k) + \alpha \sum_{j\in\mathcal{N}_i} x_j(k) + \alpha \sum_{j\in\mathcal{N}_i} w_{ij}(k).$$

Define

$$v(k) = \begin{bmatrix} \sum_{j\in\mathcal{N}_1} w_{1j}(k) \\ \vdots \\ \sum_{j\in\mathcal{N}_n} w_{nj}(k) \end{bmatrix}.$$

Therefore,

$$x(k+1) = (I - \alpha L)x(k) + \alpha w(k),$$

where we assume that $w(k)$ is i.i.d., zero mean and has a bounded second moment. The covariance of $w(k)$ is defined as Q

If α is fixed, then we cannot achieve consensus. Hence, we need to use a time varying $\alpha(k)$.

$$x(k+1) = (I - \alpha(k)L)x(k) + \alpha(k)v(k), \tag{3}$$

We choose $\alpha(k) \geq 2/(\lambda_2(L) + \lambda_n(L))$ to satisfies the following condition:

1. $\sum_{k=0}^{\infty} \alpha(k) = \infty$.
2. $\sum_{k=0}^{\infty} \alpha(k)^2 < \infty$.

Condition 2 implies that $\alpha(k) \to 0$.

One possible choice $\alpha(k) = 1/(k+1)$.

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} = \frac{\pi^2}{6} < 0.$$

In fact, we can choose $\alpha(k) = (k+1)^{-\varphi}$, for any $0.5 < \varphi < 1$.

Define $y(k) = x(k) - Jx(k)$. (Notice that this definition is different from our previous one, where $y(k) = x(k) - Jx(0)$. *why?*) Define $\theta(k) = \mathbf{1}^T x(k)/n$. Hence, $x(k) = \theta(k)\mathbf{1} + y(k)$.

By (3), we have

$$\theta(k+1) = \theta(k) + \alpha(k)\mathbf{1}^T v(k)/n.$$

Hence, for any $k_1 > k_2$,

$$\mathbb{E}(\theta(k_1) - \theta(k_2))^2 = \frac{\mathbf{1}^T Q \mathbf{1}}{n^2} \sum_{t=k_2}^{k_1-1} \alpha(t)^2.$$

Hence, $\theta(k)$ converges in L_2. Define θ as the L_2 limit of θ_k.

Now let us look at $y(k)$. By (3),

$$y(k+1) = [(I - \alpha(k)L)(I - J)]y(k) + \alpha(k)(I - J)v(k).$$

Let us define $\mathcal{P}(k) = (I - \alpha(k)L)(I - J)$. Therefore,

$$y(k+1) = \prod_{t=0}^{k} \mathcal{P}(t) y(0) + \sum_{\tau=0}^{k} \left(\prod_{t=\tau+1}^{k} \mathcal{P}(t) \right) \alpha(\tau)(I - J)v(\tau).$$

Clearly,

$$\|\mathcal{P}(k)\| = 1 - \alpha(k)\lambda_2(L).$$

Hence,

$$\left\| \prod_{t=k_1}^{k_2} \mathcal{P}(t) \right\| \le \prod_{t=k_1}^{k_2} (1 - \alpha(t)\lambda_2(L)) \le \prod_{t=k_1}^{k_2} \exp\left(-\alpha(t)\lambda_2(L)\right) = \exp\left(-\sum_{t=k_1}^{k_2} \alpha(t)\lambda_2(L) \right),$$

which implies that

$$\lim_{k\to\infty} \prod_{t=0}^{k} \mathcal{P}(k) = 0.$$

On the other hand,

$$\mathbb{E} \left\| \left(\prod_{t=\tau+1}^{k} \mathcal{P}(t) \right) \alpha(\tau)(I - J)v(\tau) \right\|^2 \le \beta \exp\left(-2 \sum_{t=\tau+1}^{k} \alpha(t)\lambda_2(L) \right) \alpha(\tau)^2.$$

where $\beta = \operatorname{tr}((I-J)Q(I-J))$. Hence

$$\mathbb{E}\left\|\sum_{\tau=0}^{k}\left(\prod_{t=\tau+1}^{k}\mathcal{P}(t)\right)\alpha(\tau)(I-J)v(\tau)\right\|^2 = \sum_{\tau=0}^{k}\mathbb{E}\left\|\left(\prod_{t=\tau+1}^{k}\mathcal{P}(t)\right)\alpha(\tau)(I-J)v(\tau)\right\|^2$$
$$\leq \beta\sum_{\tau=0}^{k}\left[\exp\left(-2\sum_{t=\tau+1}^{k}\alpha(t)\lambda_2(L)\right)\alpha(\tau)^2\right] \to 0.$$

Hence, $y(k) \to 0$. As a result, $x(k)$ converges to $\theta\mathbf{1}$ in the mean square sense (L_2).