Event-Based Estimation

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1 Problem Formulation

Let $x \in \mathbb{R}^n$ be the state, such that $x \mathcal{N}(,\Sigma)$. $y \in \mathbb{R}^m$ is the sensor measurement, where y_i is the measurement from the ith sensor, such that

$$y_i = a_i x + v_i.$$

Assume that $x, v_1, \dots, v_m$ are all linearly independent and $v_i \sim \mathcal{N}(0, r_i)$.

Let γ_i be a binary variable, such that $\gamma_i = 0$ if the sensor is not selected. $\gamma_i = 1$ if the sensor is selected.

In the previous lecture, we consider γ_i to be independent from y_i.

In this lecture, we will assume γ_i depends on y_i.

Consider the sensors employ the following strategy:

1. Each sensor randomly generates a uniformly distributed random variable ξ_i on $[0, 1]$.
2. Sensor i computes a function:

   $$g_i(y_i) = \exp\left(-\frac{y_i^2}{2\delta_i}\right),$$

   where $\delta_i > 0$ is a fixed parameter.
3. Sensor i decides whether to send the measurement or not:

   $$\gamma_i = \begin{cases} 0 & \text{if } \xi_i \le g_i(y_i) \\ 1 & \text{if } \xi_i > g_i(y_i) \end{cases}$$

Define the information set $\mathcal{I} = \{\gamma_1, \dots, \gamma_m, \gamma_1 y_1, \dots, \gamma_m y_m\}$ and state estimation and the estimation error covariance matrix as $\hat{x} \triangleq \mathbb{E}(x|\mathcal{I}), P \triangleq \mathrm{Cov}(x|\mathcal{I})$ respectively.

2 Optimal Estimator

Consider single sensor case. The joint probability of x, y, γ satisfies

$$P(x \in X, y \in Y, \gamma = 0) = P(x \in X)P(y \in Y|x \in X)P(\gamma = 0|y \in Y)$$
$$= \int_{x\in X}\int_{y\in Y} \frac{1}{\sqrt{(2\pi)^k|\Sigma|}} \exp\left(-\frac{1}{2}(x-\bar{x})^T\Sigma^{-1}(x-\bar{x})\right) \frac{1}{\sqrt{2\pi r}} \exp\left(-\frac{1}{2r}(y-ax)^2\right) g(y)dxdy.$$

Hence,

$$P(x \in X, \gamma = 0) = \alpha \int_{x\in X}\int_{y\in\mathbb{R}} \exp\left[-\frac{1}{2}\left((x-\bar{x})^T\Sigma^{-1}(x-\bar{x}) + \frac{1}{r}(y-ax)^2 + \frac{1}{\delta}y^2\right)\right] dxdy$$
$$= \alpha \int_{x\in X} \exp\left[-\frac{1}{2}(x-\bar{x})^T\Sigma^{-1}(x-\bar{x})\right] \int_{y\in R} \exp\left[-\frac{1}{2}\left(\frac{1}{r}(y-ax)^2 + \frac{1}{\delta}y^2\right)\right] dydx$$

Since

$$\begin{aligned}\frac{1}{r}(y-ax)^2 + \frac{1}{\delta}y^2 &= \left(\frac{1}{r}+\frac{1}{\delta}\right)y^2 - \frac{2}{r}(ax)y + \frac{1}{r}(ax)^2 \\ &= \left[\sqrt{\frac{r+\delta}{r\delta}}\, y - \sqrt{\frac{\delta}{r(r+\delta)}}\,(ax)\right]^2 + \frac{1}{r+\delta}x^Ta^Tax.\end{aligned}$$

Hence,

$$P(x \in X, \gamma = 0) = \beta \int_{x\in X} \exp\left[-\frac{1}{2}\left((x-\bar{x})^T\Sigma^{-1}(x-\bar{x}) + \frac{1}{r+\delta}x^Ta^Tax\right)\right] dx$$
$$= \lambda \int_{x\in X} \exp\left[-\frac{1}{2}(x-\tilde{x})\tilde{P}^{-1}(x-\tilde{x})\right],$$

where

$$\tilde{P} = \left(\Sigma^{-1} + \frac{a^Ta}{r+\delta}\right)^{-1} = \Sigma - \Sigma a^T(a\Sigma a^T + r + \delta)^{-1}a\Sigma.$$

and

$$\tilde{x} = P\Sigma^{-1}\bar{x} = \bar{x} + \Sigma a^T(a\Sigma a^T + r + \delta)^{-1}(0 - a\bar{x}).$$

Hence, if $P(x|\gamma = 0)$ is Gaussian distributed with mean $\tilde{x}$ and covariance P.

On the other hand,

$$P(x \in X|y \in Y, \gamma = 1) = \frac{P(\gamma = 1|x \in X, y \in Y)P(x \in X|y \in Y}{P(\gamma = 1|y \in Y)} = \frac{(1-g(y))P(x \in X|y \in Y)}{1-g(y)}$$
$$= P(x \in X|y \in Y).$$

Hence, $P(x|y, \gamma = 1)$ is a Gaussian distribution with mean $\bar{x} + \Sigma a^T(a\Sigma a^T + r)^{-1}(y - a\bar{x})$ and variance $P = (\Sigma^{-1} + a^Ta/r)^{-1}$.

Therefore

$$\hat{x} = \bar{x} + \Sigma a^T(a\Sigma a^T + r + (1-\gamma)\delta)^{-1}(\gamma y - a\bar{x}),$$

and

$$P = \left(\Sigma^{-1} + \frac{a^T a}{r + (1-\gamma)\delta} \right)^{-1}$$

For multi-sensors, define

$$\Gamma = diag(\gamma_1, \ldots, \gamma_m),\ \Delta = diag(\delta_1, \ldots, \delta_m).$$

then

$$\hat{x} = \bar{x} + \Sigma C^T (C\Sigma C^T + R + (I-\Gamma)\Delta)^{-1}(\Gamma y - C\bar{x}),$$

and

$$P = \left(\Sigma^{-1} + \sum_{i=1}^{m} \frac{a_i^T a_i}{r_i + (1-\gamma_i)\delta_i} \right)^{-1} = \Sigma - \Sigma C^T (C\Sigma C^T + R + (I-\Gamma)\Delta)^{-1} C\Sigma.$$

3 Communication Rate

Consider single sensor case and assume that $\bar{x} = 0$ Let us define the communication rate $\lambda = P(\gamma = 1)$. Then

$$P(\gamma = 0) = \mathbb{E}\ P(\xi \le g(y)|y) = \mathbb{E}g(y).$$

y is Gaussian with mean 0 and variance $r + a\Sigma a^T$. Hence,

$$\begin{aligned} \mathbb{E}g(y) &= \frac{1}{\sqrt{2\pi(r + a\Sigma a^T)}} \int_{y\in\mathbb{R}} \exp\left[-\frac{1}{2} \left(\frac{y^2}{r + a\Sigma a^T} + \frac{y^2}{\delta} \right) \right] \\ &= \frac{1}{\sqrt{1 + \frac{r + a\Sigma a^T}{\delta}}}. \end{aligned}$$

For multi sensor case,

$$\lambda_i = 1 - \frac{1}{\sqrt{1 + \frac{r_i + a_i \Sigma a_i^T}{\delta_i}}}.$$