Lecture 5: Control Over Lossy Networks

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1 Classical LQG Control

The system:

$$x_{k+1} = Ax_k + Bu_k + w_k,$$
$$y_k = Cx_k + v_k$$

$x_0 \sim \mathcal{N}(0, \Sigma)$, $w_k \sim \mathcal{N}(0, Q)$, $v_k \sim \mathcal{N}(0, R)$.

Information available for the controller at time k:

$$Y_k = (y_0, \dots, y_k).$$

The control at time k is a function of the information Y_k: $u_k(Y_k)$.

The goal of a finite horizon LQG problem is to find a controller that minimizes the following quadratic cost:

$$J(N) = \min_{u_0,\dots,u_N} \mathbb{E} \sum_{k=0}^{N} \left(x_k^T W x_k + u_k^T U u_k \right).$$

1.1 Optimal Estimator Design

Since the system is linear, the following Kalman filtering equations holds:

1. **Initialization:**
$$\hat{x}_{0|-1} = 0,\ P_{0|-1} = \Sigma. \tag{1}$$
2. **Prediction:**
$$\hat{x}_{k+1|k} = A\hat{x}_k + Bu_k,\ P_{k+1|k} = AP_kA^T + Q. \tag{2}$$
3. **Correction:**
$$\hat{x}_{k+1} = \hat{x}_{k+1|k} + P_{k+1|k}C^T(CP_{k+1|k}C^T + R)^{-1}(y_{k+1} - C\hat{x}_{k+1|k}), \tag{3}$$
$$P_{k+1} = P_{k+1|k} - P_{k+1|k}C^T(CP_{k+1|k}C^T + R)^{-1}CP_{k+1|k}. \tag{4}$$

And we have that

$$\mathbb{E}(x_k|Y_k) = \hat{x}_k,\ \mathrm{Cov}(x_k|Y_k) = P_k.$$

One important thing to notice: the P_k is independent from u_k. This is because the system is linear and hence we can subtract u_k.

1.2 Optimal Controller Design and Separation Principle

Now we can try to solve the optimal control design problem by dynamical programming.

Define the value function

$$V(t) = \min_{u_t,\ldots,u_N} \mathbb{E}\left[\sum_{k=t}^{N}\left(x_k^T W x_k + u_k^T U u_k\right)\right]$$

Clearly

$$J(N) = V(0),$$

and

$$V(N) = \mathbb{E}x_N^T W x_N.$$

Now, by Bellman equation

$$V(t) = \min_{u_t} \mathbb{E}\left(x_t^T W x_t + u_t^T U u_t + V(t+1)\right) \tag{5}$$

We will guess that

$$V(t) = \mathbb{E}x_t^T S_t x_t + c_t \tag{6}$$

To prove this, we will use induction.

Clearly $S_N = W$ and $c_N = 0$.

Now suppose (6) holds for $t+1$, and we look at the following quantity:

$$\begin{aligned}
&\mathbb{E}\left(x_t^T W x_t + u_t^T U u_t + V(t+1)\right)\\
&\quad = \mathbb{E}x^T(W + A^T S_{t+1} A)x + \mathrm{tr}(S_{t+1}Q) + c_{t+1}\\
&\quad + \mathbb{E}\left[u_t^T(U + B^T S_{t+1} B)u_t + x_t^T A^T S_{t+1} B u_t + u_t^T B^T S_{t+1} A x_t\right]
\end{aligned}$$

Notice that the controller do not know x_t. Hence, let us rewrite x_t as

$$x_t = \hat{x}_t + x_t - \hat{x}_t = \hat{x}_t + e_t.$$

Theorem 1. *1. e_k is independent of Y_k and hence $\hat{x}_k$ and u_k.*

2.

$$\mathbb{E}\hat{x}_k^T S \hat{x}_k = \mathbb{E}x_k^T S x_k - \mathrm{tr}(SP_k)$$

Proof. 1. Notice that

$$\mathbb{E}(e_k|Y_k) = \mathbb{E}(x_k|Y_k) - \mathbb{E}(\hat{x}_k|Y_k) = \hat{x}_k - \hat{x}_k = 0.$$

Hence, e_k is linearly independent from Y_k. Since e_k and Y_k are jointly Gaussian, e_k and Y_k are independent.

2. Since

$$\mathbb{E}x_k^T S x_k = \mathbb{E}\hat{x}_k^T S \hat{x}_k + \mathbb{E}\hat{x}_k^T S e_k + \mathbb{E}e_k^T S \hat{x}_k + \mathbb{E}e_k^T S e_k = \mathbb{E}\hat{x}_k^T S \hat{x}_k + 0 + 0 + \mathrm{tr}(SP_k)$$

□

Now let us look at

$$
\begin{aligned}
&\mathbb{E}\left[u_t^T(U+B^TS_{t+1}B)u_t + x_t^TA^TS_{t+1}Bu_t + u_t^TB^TS_{t+1}Ax_t\right] \\
&\quad = \mathbb{E}\left[u_t^T(U+B^TS_{t+1}B)u_t + \hat{x}_t^TA^TS_{t+1}Bu_t + u_t^TB^TS_{t+1}A\hat{x}_t\right] \\
&\quad = \mathbb{E}\left[(u_t-u_t^*)^T(U+B^TS_{t+1}B)(u_t-u_t^*) - \hat{x}_t^TA^TS_{t+1}B(U+B^TS_{t+1}B)^{-1}B^TS_{t+1}A\hat{x}_t\right]
\end{aligned}
$$

where $u_t^* = -(U+B^TS_{t+1}B)^{-1}B^TS_{t+1}A\hat{x}_t$. Hence

$$
\begin{aligned}
V(t) = {} & \mathbb{E}x_t^T(W + A^TS_{t+1}A - A^TS_{t+1}B(U+B^TS_{t+1}B)^{-1}B^TS_{t+1}A)x_t \\
& + c_{t+1} + \mathrm{tr}(A^TS_{t+1}B(U+B^TS_{t+1}B)^{-1}B^TS_{t+1}AP_k) + \mathrm{tr}(S_{t+1}Q)
\end{aligned}
$$

Therefore

$$
S_t = W + A^TS_{t+1}A - A^TS_{t+1}B(U+B^TS_{t+1}B)^{-1}B^TS_{t+1}A, \tag{7}
$$

and

$$
c_t = c_{t+1} + \mathrm{tr}(A^TS_{t+1}B(U+B^TS_{t+1}B)^{-1}B^TS_{t+1}AP_k) + \mathrm{tr}(S_{t+1}Q).
$$

Thus,

$$
J(N) = \mathbb{E}(x_0^TS_0x_0) + c_0 = \mathrm{tr}(S_0\Sigma) + c_0.
$$

1.3 Infinite Horizon LQG problem

Define J as

$$
J = \lim_{N\to\infty} \frac{J(N)}{N}.
$$

We consider the problem of finding a controller that minimizes the infinite horizon cost J.

Notice that (7) is a Riccati equation. Hence, if $N \to \infty$, then S_k converges to S, which is the fixed solution of

$$
S = W + A^TSA - A^TSB(U+B^TSB)^{-1}B^TSA, \tag{8}
$$

The optimal controller is given by

$$
u_k^* = -(U+B^TSB)^{-1}B^TSA\hat{x}_k.
$$

2 Witsenhausen's Counterexample

Consider $x_0 \sim \mathcal{N}(0,\sigma^2)$.

1. The first player knows x_0 and he computes an

$$
x_1 = f(x_0),
$$

which is a function of x_0.

2. The first player sends x_1 to the second player though a noisy channel. Therefore, the second player receives

$$y_2 = x_1 + v,$$

where $v \sim \mathcal{N}(0,1)$.

3. The second player then computes $x_2 = g(y_2)$.

The goal is to minimize the following cost function

$$J = \min_{f,g} \ \mathbb{E}\, k^2(x_0 - x_1)^2 + (x_1 - x_2)^2$$

Alternatively, one can consider the following equivalent scheme:

1. The controller knows x_0 and it computes an control u

$$u = f(x_0),$$

which is a function of x_0.

2. The state of the system satisfies the following update equation:

$$x_1 = x_0 + u.$$

3. The second player observe the system via a noisy sensor:

$$y_2 = x_1 + v,$$

where $v \sim \mathcal{N}(0,1)$.

4. The second player then computes the state estimate $x_2 = g(y_2)$.

The goal is to minimize the following cost function

$$J = \min_{f,g} \ \mathbb{E}\, k^2 u^2 + (x_1 - x_2)^2$$

2.1 Optimal linear strategy

We adopt the first setting. Consider that both $f(x) = \lambda x$ and $g(x) = \mu x$ are linear, then

$$J = \min_{\lambda,\mu} \mathbb{E}\, k^2(1-\lambda)^2 x_0^2 + (\lambda x_0 - \mu(\lambda x_0 + v))^2$$

Therefore,

$$\mu = \frac{\lambda^2\sigma^2}{1+\lambda^2\sigma^2},$$

and

$$\lambda = \arg\min_{\lambda} \ k^2\sigma^2(1-\lambda)^2 + \frac{\lambda^2\sigma^2}{1+\lambda^2\sigma^2}.$$

If $k^2\sigma^2 = 1$ and $k \to 0$, then $\lambda \approx 1$ and $J \approx 1$.

3 Nonlinear strategy

One can prove that for small k and $k^2\sigma^2 = 0$, the following design is better than the linear design:

$$f(x) = \sigma \mathrm{sgn}(x),\; g(x) = \sigma \frac{1 - e^{-2\sigma x}}{1 + e^{-2\sigma x}}.$$

4 Control Over Lossy Networks

The system:

$$\begin{aligned} x_{k+1} &= Ax_k + \nu_k B u_k + w_k, \\ y_k &= Cx_k + v_k \end{aligned}$$

$x_0 \sim \mathcal{N}(0, \Sigma)$, $w_k \sim \mathcal{N}(0, Q)$, $v_k \sim \mathcal{N}(0, R)$.

ν_k is an i.i.d. Bernouli process with $P(\nu_k = 1) = \lambda$.

The goal of a finite horizon LQG problem is to find a controller that minimizes the following quadratic cost:

$$J = \min_{u_0, \ldots, u_N} \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \sum_{k=0}^{N} \left(x_k^T W x_k + \nu_k u_k^T U u_k \right).$$

4.1 TCP case

Information available for the controller at time k:

$$\mathcal{I}_k = (y_0, \ldots, y_k, \nu_0, \ldots, \nu_k).$$

J is finite if and only if the following Riccati equation has a positive semidefinite solution:

$$S = W + A^T S A - \lambda A^T S B (U + B^T S B)^{-1} B^T S A, \tag{9}$$

Optimal Filter:

1. **Initialization:**
$$\hat{x}_{0|-1} = 0,\; P_{0|-1} = \Sigma. \tag{10}$$

2. **Prediction:**
$$\hat{x}_{k+1|k} = A\hat{x}_k + \nu_k B u_k,\; P_{k+1|k} = A P_k A^T + Q. \tag{11}$$

3. **Correction:**
$$\hat{x}_{k+1} = \hat{x}_{k+1|k} + P_{k+1|k} C^T (C P_{k+1|k} C^T + R)^{-1} (y_{k+1} - C\hat{x}_{k+1|k}), \tag{12}$$
$$P_{k+1} = P_{k+1|k} - P_{k+1|k} C^T (C P_{k+1|k} C^T + R)^{-1} C P_{k+1|k}. \tag{13}$$

Optimal Control:
The optimal controller is given by

$$u_k^* = -(U + B^T SB)^{-1} B^T SA\hat{x}_k,$$

where S is the solution of (9).

4.2 UDP case

Information available for the controller at time k:

$$Y_k = (y_0, \ldots, y_k).$$

We do not know whether u_k has been applied to the system or not. The control actually affect the estimation performance. The optimal control law and the stability of the system is unknown.