Lecture 3: Functions of Symmetric Matrices

Yilin Mo

July 2, 2015

1 Recap

1. *Bayes Estimator:*

   (a) **Initialization:**
   $$f(x_0|Y_{-1}) = f(x_0).$$

   (b) **Correction:**
   $$f(x_k|Y_k) = \alpha f(y_k|x_k) f(x_k|Y_{k-1}),$$
   where
   $$\alpha = \left(\int_{\mathbb{R}^n} f(y_k|x_k) f(x_k|Y_{k-1}) \,\mathrm{d}\, x_k\right)^{-1}.$$
   The MMSE estimation can be derived as
   $$\hat{x} = \mathbb{E}(x_k|Y_k) = \int_{\mathbb{R}^n} x_k f(x_k|Y_k) \,\mathrm{d}\, x_k.$$

   (c) **Prediction:**
   $$f(x_{k+1}|Y_k) = \int_{\mathbb{R}^n} f(x_{k+1}|x_k) f(x_k|Y_k) \,\mathrm{d}\, x_k.$$

2. *Kalman Filter:*

   (a) **Initialization:**
   $$\hat{x}_{0|-1} = 0,\ P_{0|-1} = \Sigma. \tag{1}$$

   (b) **Prediction:**
   $$\hat{x}_{k+1|k} = A\hat{x}_{k|k},\ P_{k+1|k} = AP_{k|k}A^T + Q. \tag{2}$$

   (c) **Correction:**
   $$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + P_{k+1|k}C^T(CP_{k+1|k}C^T + R)^{-1}(y_{k+1} - C\hat{x}_{k+1|k}), \tag{3}$$
   $$P_{k+1|k+1} = P_{k+1|k} - P_{k+1|k}C^T(CP_{k+1|k}C^T + R)^{-1}CP_{k+1|k}. \tag{4}$$

3. Linear Estimator:

   (a) Initialization:
   $$\hat{x}_{0|-1} = 0.$$

   (b) Prediction:
   $$\hat{x}_{k+1|k} = A\hat{x}_{k|k}.$$

   (c) Correction:
   $$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left(y_{k+1} - C\hat{x}_{k+1|k}\right).$$

   Estimation error covariance of the linear filter satisfies:
   $$\begin{aligned} P_{0|-1} &= \Sigma,\ P_{k+1|k} = AP_{k|k}A^T + Q, \\ P_{k+1|k+1} &= (I - K_{k+1}C)P_{k+1|k}(I - K_{k+1}C)^T + K_{k+1}RK_{k+1}. \end{aligned}$$

2 Kalman Filtering with Intermittent Observations: Problem Formulation

Suppose the sensor send its measurements through an erasure channel:

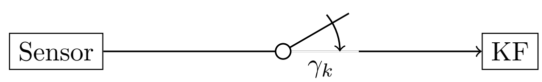

Figure 1: Kalman Filtering with Intermittent Observations

Let γ_k be a binary variable, such that $\gamma_k = 0$ implies that the KF does not receive y_k and $\gamma_k = 1$ implies that the KF receives y_k.

We assume that γ_k is an i.i.d. Bernoulli random variable with $P(\gamma_k = 1) = \lambda$, which is independent from $x_0, \{w_k\}, \{v_k\}$.

Hence, the information that the KF has at time k is
$$\gamma_0, \ldots, \gamma_k, \gamma_0 y_0, \ldots, \gamma_k y_k.$$

The optimal estimator is a time varying KF:

1. **Initialization:**
$$\hat{x}_{0|-1} = 0,\ P_{0|-1} = \Sigma. \tag{5}$$

2. **Prediction:**
$$\hat{x}_{k+1|k} = A\hat{x}_{k|k},\ P_{k+1|k} = AP_{k|k}A^T + Q. \tag{6}$$

3. **Correction:**

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + \gamma_{k+1} P_{k+1|k} C^T (C P_{k+1|k} C^T + R)^{-1} (y_{k+1} - C\hat{x}_{k+1|k}), \tag{7}$$

$$P_{k+1|k+1} = P_{k+1|k} - \gamma_{k+1} P_{k+1|k} C^T (C P_{k+1|k} C^T + R)^{-1} C P_{k+1|k}. \tag{8}$$

To simplify notations, we define

$$P_k \triangleq P_{k|k}.$$

Furthermore, define

$$h(X) \triangleq AXA^T + Q,\ g(X) \triangleq h(X) - h(X)C^T (Ch(X)C^T + R)^{-1} Ch(X).$$

As a result,

$$P_k = \begin{cases} h(P_{k-1}) & \text{if } \gamma_k = 0 \\ g(P_{k-1}) & \text{if } \gamma_k = 1 \end{cases}$$

h is called a Lyapunov equation and g is called a discrete-time algebraic Riccati equation.

3 Properties of Discrete-time Algebraic Riccati Equation

3.1 Symmetric Matrix

Let $\mathbb{S}^n$ be the space of real symmetric n by n matrices. $\mathcal{S}^n$ is a linear space with dimension $n(n+1)/2$.

Definition 1. *$\mathbb{S}^n_+ \subset \mathbb{S}^n$ is the set of all positive semidefinite matrices. $\mathcal{S}^n_{++} \subset \mathbb{S}^n$ is the set of all positive definite matrices.*

1. For any $X, Y \in \mathbb{S}^n_+$, $\alpha, \beta \geq 0$, $\alpha X + \beta Y \in \mathbb{S}^n_+$. $\mathbb{S}^n_+$ is a convex cone.
2. $\mathbb{S}^n_+ \bigcap \left(-\mathbb{S}^n_+\right) = \{0\}$.

$\mathbb{S}^n_+$ induces a partial order on $\mathbb{S}^n$:

$$X \geq Y \implies X - Y \in \mathbb{S}^n_+.$$

1. $0 \in \mathbb{S}^n_+ \implies X \geq X$.
2. $\mathbb{S}^n_+ \bigcap \left(-\mathbb{S}^n_+\right) = \{0\}$ implies that if $X \geq Y$ and $Y \geq X$, then $X = Y$.
3. Convexity implies that if $X \geq Y$ and $Y \geq Z$, then $X \geq Z$.

However, it is not a total order:

$$X = 0,\, Y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Neither $X \geq Y$ nor $Y \geq X$.

Theorem 1. *If the sequence $\{X_k\}$ is monotonically increasing, i.e., $X_{k+1} \geq X_k$, and there exists an M, such that for all k, $X_k \leq M$, then the following entrywise limit is well-defined*

$$\lim_{k\to\infty} X_k = X.$$

Proof.

- *Diagonal Elements:*

  $X_{k+1}(i,i) \geq X_k(i,i)$ implies that the diagonal element $X_{k+1}(i,i) \geq X_k(i,i)$. Hence, $X_k(i,i)$ is increasing and is bounded by $M(i,i)$. Therefore $X_k(i,i)$ converges.

- *Off-diagonal Elements:*

  Consider $k_1 \geq k_2$, then $X_{k_1} \geq X_{k_2}$, which implies that all principal minor is non-negative, i.e.,

  $$|X_{k_1}(i,j) - X_{k_2}(i,j)|^2 \leq |X_{k_1}(i,i) - X_{k_2}(i,i)||X_{k_1}(j,j) - X_{k_2}(j,j)|$$

  Use Cauchy Criterion to prove that the off-diagonal elements also converge.

□

3.2 Functions on $\mathbb{S}^n$

Definition 2. *A function $f : \mathbb{S}^n \to \mathbb{S}^n$ is monotonically increasing if for any $X \geq Y$, $f(X) \geq f(Y)$. A function f is decreasing if $-f$ is increasing.*

Definition 3. *A function $f : \mathbb{S}^n \to \mathbb{S}^n$ is convex if for any X, Y and $\alpha, \beta > 0, \alpha + \beta = 1$, the following inequality holds*

$$\alpha f(X) + \beta f(Y) \geq f(\alpha X + \beta Y).$$

A function f is concave if $-f$ is convex.

Some functions:

1. Affine function:

   $$h(X) = AXA^T + Q.$$

   $h(X)$ is increasing, convex and concave.

2. Inverse function:

   $$f(X) = X^{-1}.$$

   $f(X)$ is decreasing and convex on $\mathbb{S}^n_{++}$.

Proof. Consider $X, Y \in \mathbb{S}^n_{++}$. There exists an orthogonal matrix Q_1, such that

$$Q_1 X Q_1^T = \Lambda_X,$$

where Λ_X is a diagonal matrix. Define $\Lambda_X^{1/2}$ as the square root of Λ_X. Hence,

$$Q_1 \Lambda_X Q_1^T \times Q_1 \Lambda_X Q_1^T = X.$$

Let $X^{1/2} = Q_1 \Lambda_X^{1/2} Q_1^T$. Then there exists another orthogonal matrix Q_2, such that

$$Q_2 X^{-1/2} Y X^{-1/2} Q_2^T = \Lambda_Y,$$

On the other hand

$$Q_2 X^{-1/2} X X^{-1/2} Q_2^T = I.$$

The proof can be done by using the matrix $Q_2 X^{-1/2}$ to diagonalize both X and Y and use the fact that $1/x$ is decreasing and concave on $\mathbb{R}^+$ $\square$

3. Discrete-time algebraic Riccati equation:

Matrix Inversion Lemma:

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}. \tag{9}$$

Therefore,

$$g(X) = \left[(h(X))^{-1} + C^T R^{-1} C\right]^{-1}.$$

$g(X)$ is increasing, concave and non-negative on $\mathbb{S}^n_+$. (**why?**)

Another way of thinking:

Consider the update equation of a linear filter:

$$\begin{aligned}\varphi(X, K) &= (I - KC)h(X)(I - KC)^T + KRK^T \\ &= K(Ch(X)C^T + R)K^T - KCh(X) - h(X)C^T K^T + h(X).\end{aligned}$$

Define $K^* = h(X)C^T(Ch(X)C^T + R)^{-1}$, then

$$\varphi(X, K) = g(X) + (K - K^*)(Ch(X)C^T + R)(K - K^*)^T$$

Thus

$$g(X) = \min_K \varphi(X, K).$$

Fix K, $\varphi(X, K)$ is increasing and affine. Thus, $g(X)$ is increasing, concave and non-negative on $\mathbb{S}^n_+$. (**why?**)